

# Non-Commutative Complete Mellin Representation for Feynman Amplitudes

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## Abstract

We extend the complete Mellin (CM) representation of Feynman amplitudes to the non-commutative quantum field theories. This representation is a versatile tool. It provides a quick proof of meromorphy of Feynman amplitudes in parameters such as the dimension of space-time. In particular it paves the road for the dimensional renormalization of these theories. This complete Mellin representation also allows the study of asymptotic behavior under rescaling of arbitrary subsets of external invariants of any Feynman amplitude.

## 1 Introduction

Recently non-commutative quantum field theories such as the  $\phi^{*4}$  and the Gross-Neveu<sub>2</sub> models have been shown renormalizable [1, 2, 3, 4, 5, 6] provided the propagator is modified to obey Langmann-Szabo duality [7]. These theories are called “vulcanized”, and in [8, 9] the Feynman-Schwinger parametric representation was extended to them. For recent reviews, see [10, 11].

In this paper we perform a further step, generalizing to these vulcanized NCQFT the complete Mellin (CM) representation. This CM representation is derived from the parametric representation established in [8, 9] and provides a starting point for the study of dimensional renormalization and of the asymptotic behavior of Feynman amplitudes under arbitrary rescaling of external invariants.

The study of asymptotic behaviors in conventional (commutative) field theories started in the 1970’s with the papers [12, 13, 14] which use the BPHZ renor-

malization scheme. Their approach is based on the Feynman-Schwinger parametric representation of amplitudes. This representation involves “Symanzik” or “topological” polynomials associated to the Feynman diagram. In commutative theories these “topological polynomials” are written as sums over the spanning trees or 2-trees of the diagram [15, 16]. The corresponding mathematical theory goes back to the famous tree matrix theorem of Birkhoff (see [17] and references therein).

In vulcanized NCQFTs, propagators are no longer based on the heat kernel but on the Mehler kernel. The kernel being still quadratic in position space, explicit integration over all space variables is still possible. It leads to the NC-QFT parametric representation. This representation no longer involves ordinary polynomials in the Schwinger parameters, but hyperbolic polynomials [8]. As diagrams in NCQFT are ribbon graphs, these hyperbolic polynomials contain richer topological information than in the commutative case. In particular they depend on the genus of the Riemann surface on which the graphs are defined.

Returning to the Mellin transform technique, it was introduced in commutative field theory to prove theorems on the asymptotic expansion of Feynman diagrams. The general idea was to prove the existence of an asymptotic series in powers of  $\lambda$  and powers of logarithms of  $\lambda$ , under rescaling by  $\lambda$  of some (Euclidean) external invariants associated to the Feynman amplitudes. The Mellin transform with respect to the scaling parameter allows to obtain this result in some cases and to compute the series coefficients, even for renormalized diagrams [18]. But it did not work for *arbitrary* subsets of invariants.

In the subsequent years, the subject advanced further. The rescaling of internal squared masses (in order to study the infrared behavior of the amplitudes) was treated in [20]. There the concept of “FINE” polynomials was introduced (that is, those being factorizable in each Hepp sector [19] of the variables<sup>1</sup>). It was then argued that the Mellin transform may be “desingularized”, which means that the integrand of the inverse Mellin transform (which gives back the Feynman amplitude as a function of  $\lambda$ ) has a meromorphic structure, so that the residues of its various poles generate the asymptotic expansion in  $\lambda$ . However, this is not the case under arbitrary rescaling, because in many diagrams the FINE property simply does not occur.

A first solution to this problem was presented in [20] by introducing the so-called “multiple Mellin” representation, which consists in splitting the Symanzik polynomials in a certain number of pieces, each one of which having the FINE property. Then, after scaling by the parameter  $\lambda$ , an asymptotic expansion can be obtained as a sum over all Hepp sectors. This is always possible if one adopts, as done in [21, 22, 23], the extreme point of view to split the Symanzik polynomials in all its monomials. Moreover, this apparent complication is compensated by the fact that one can dispense with the use of Hepp sectors altogether. This is called the “complete Mellin” (CM) representation.

The CM representation provides a general proof of the existence of an asymptotic expansion in powers and powers of logarithms of the scaling parameter in

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<sup>1</sup>A Hepp sector is a complete ordering of the Schwinger parameters.

the most general case. Moreover the integrations over the Schwinger parameters can be explicitly performed, and we are left with the pure geometrical study of convex polyhedra in the Mellin variables. The results of [20] are obtained in a simpler way [21], and asymptotic expansions are computed in a more compact form, without any division of the integral into Hepp sectors.

Moreover the CM representation allows a unified treatment of the asymptotic behavior of both ultraviolet convergent and divergent renormalized amplitudes. Indeed, as shown in [21, 22], the renormalization procedure does not alter the algebraic structure of integrands in the CM representation. It only changes the set of relevant integration domains in the Mellin variables. The method allows the study of dimensional regularization [22, 23] and of the infrared behavior of amplitudes relevant to critical phenomena [24].

The CM representation is up to now the only tool which provides asymptotic expansions of Feynman amplitudes in powers and powers of logarithms in the most general rescaling regimes. More precisely consider a Feynman amplitude  $G(s_k)$  written in terms of its external invariants  $s_k$  (including particle masses), and an asymptotic regime defined by

$$s_k \rightarrow \lambda^{a_k} s_k, \quad (1)$$

where  $a_k$  may be positive, negative, or zero. Let  $\lambda$  go to infinity (in this way both ultraviolet and infrared behaviors are treated on the same footing). Then  $G(s_k)$  as a function of  $\lambda$  has an asymptotic expansion of the form:

$$G(\lambda; s_k) = \sum_{p=p_{\max}}^{-\infty} \sum_{q=0}^{q_{\max}(p)} G_{pq}(s_k) \lambda^p \ln^q \lambda, \quad (2)$$

where  $p$  runs over decreasing rational values, with  $p_{\max}$  as leading power, and  $q$ , for a given  $p$ , runs over a finite set of non-negative integer values.

The CM representation proves this result for any commutative field theory, including gauge theories in arbitrary gauges [25].

The above general result should hold *mutatis mutandis* for non-commutative field theories. In this paper we provide a starting point for this analysis by constructing the CM representation for  $\phi^{*4}$  (Theorem 4.1 below). Moreover using this representation we check meromorphy of the Feynman amplitudes in the dimension of space-time (Theorem 5.1).

The main difference with the commutative case is that this integral representation (previously true in the sense of *functions* of the external invariants) now holds only in the sense of *distributions*. Indeed the distributional character of commutative amplitudes (in momentum space) reduces to a single overall  $\delta$ -function of momentum conservation. This is no longer true for vulcanized NCQFT amplitudes, which must be interpreted as distributions to be smeared against test functions of the external variables.

## 2 Commutative CM representation

For simplicity we consider a scalar Feynman amplitude. To get the CM representation [21], rewrite the Symanzik polynomials as

$$U(\alpha) = \sum_j \prod_{\ell=1}^L \alpha_\ell^{u_{\ell j}} \equiv \sum_j U_j, \quad V(\alpha) = \sum_k s_k \left( \prod_{\ell=1}^L \alpha_\ell^{v_{\ell k}} \right) \equiv \sum_k V_k, \quad (3)$$

where  $j$  runs over the set of spanning trees and  $k$  over the set of the 2-trees,

$$u_{\ell j} = \begin{cases} 0 & \text{if the line } \ell \text{ belongs to the 1-tree } j \\ 1 & \text{otherwise} \end{cases} \quad (4)$$

and

$$v_{\ell k} = \begin{cases} 0 & \text{if the line } \ell \text{ belongs to the 2-tree } k \\ 1 & \text{otherwise.} \end{cases} \quad (5)$$

The Mellin transform relies on the fact that for any function  $f(u)$ , piecewise smooth for  $u > 0$ , if the integral

$$g(x) = \int_0^\infty du u^{-x-1} f(u) \quad (6)$$

is absolutely convergent for  $\alpha < \text{Re } x < \beta$ , then for  $\alpha < \sigma < \beta$

$$f(u) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} dx g(x) u^x. \quad (7)$$

Consider  $D$  the space-time dimension to be for the moment real positive. Taking  $f(u) = e^{-u}$ , and applying to  $u = V_k/U$  one gets

$$e^{-V_k/U} = \int_{\tau_k}^\infty \Gamma(-y_k) \left( \frac{V_k}{U} \right)^{y_k}, \quad (8)$$

where  $\int_{\tau_k}$  is a short notation for  $\int_{-\infty}^{+\infty} \frac{d(\text{Im } y_k)}{2\pi}$ , with  $\text{Re } y_k$  fixed at  $\tau_k < 0$ . We may now recall the identity

$$\Gamma(u) (A+B)^{-u} = \int_{-\infty}^\infty \frac{d(\text{Im } x)}{2\pi} \Gamma(-x) A^x \Gamma(x+u) B^{-x-u}. \quad (9)$$

Taking  $A \equiv U_1(x)$  and  $B \equiv U_2 + U_3 + \dots$  and using iteratively the identity above, leads, for  $u = \sum_k y_k + D/2$ , to

$$\Gamma \left( \sum_k y_k + \frac{D}{2} \right) U^{-\sum_k y_k - \frac{D}{2}} = \int_\sigma \prod_j \Gamma(-x_j) U_j^{x_j}, \quad (10)$$

with  $\text{Re } x_j = \sigma_j < 0$ ,  $\text{Re } (\sum_k y_k + \frac{D}{2}) = \sum_k \tau_k + \frac{D}{2} > 0$ , and  $\int_\sigma$  means  $\int_{-\infty}^{+\infty} \prod_j \frac{d(\text{Im } x_j)}{2\pi}$  with  $\sum_j x_j + \sum_k y_k = -\frac{D}{2}$ . Then, using (8) and (10), the amplitude is written as

$$I_{\mathcal{G}}(s_k, m_l^2) = \int_\Delta \frac{\prod_j \Gamma(-x_j)}{\Gamma(-\sum_j x_j)} \prod_k s_k^{y_k} \Gamma(-y_k) \int_0^\infty \prod_l d\alpha_l \alpha_l^{\phi_l-1} e^{-\sum_l \alpha_l m_l^2}, \quad (11)$$

where

$$\phi_i \equiv \sum_j u_{ij} x_j + \sum_k v_{ik} y_k + 1. \quad (12)$$

The symbol  $\int_{\Delta}$  means integration over the independent variables  $\frac{\text{Im } x_j}{2\pi}, \frac{\text{Im } y_k}{2\pi}$  in the convex domain  $\Delta$  defined by ( $\sigma$  and  $\tau$  standing respectively for  $\text{Re } x_j$  and  $\text{Re } y_k$ ):

$$\Delta = \left\{ \sigma, \tau \left| \begin{array}{l} \sigma_j < 0; \tau_k < 0; \sum_j x_j + \sum_k y_k = -\frac{D}{2}; \\ \forall i, \text{Re } \phi_i \equiv \sum_j u_{ij} \sigma_j + \sum_k v_{ik} \tau_k + 1 > 0 \end{array} \right. \right\}. \quad (13)$$

This domain  $\Delta$  is non-empty as long as  $D$  is positive and small enough so that every subgraph of  $G$  has convergent power counting [21], hence in particular for the  $\phi^4$  theory it is always non empty for any graph for  $0 < D < 2$ .

The  $\alpha$  integrations may be performed, using the well-known representation for the gamma function, so that we have

$$\int_0^\infty d\alpha_l e^{-\alpha_l m_l^2} \alpha_l^{\phi_l - 1} = \Gamma(\phi_l) (m_l^2)^{-\phi_l} \quad (14)$$

and we finally get the CM representation of the amplitude in the scalar case:

$$I_G(s_k, m_l^2) = \int_{\Delta} \frac{\prod_j \Gamma(-x_j)}{\Gamma(-\sum_j x_j)} \prod_k s_k^{y_k} \Gamma(-y_k) \prod_l (m_l^2)^{-\phi_l} \Gamma(\phi_l). \quad (15)$$

This representation can now be extended to complex values of  $D$ . For instance for a massive  $\phi^4$  graph, it is analytic in  $D$  for  $\Re D < 2$ , and meromorphic in  $D$  in the whole complex plane with singularities at rational values; furthermore its dimensional analytic continuation has the same unchanged  $CM$  integrand but translated integration contours [22, 23].

### 3 Non-commutative parametric representation

Let us summarize the results of [8]. Define the antisymmetric matrix  $\sigma$  as

$$\sigma = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} \text{ with } \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \quad (16)$$

The  $\delta$ -functions appearing in the vertices can be rewritten as an integral over some new variables  $p_V$  called *hypermomenta*, via the relation

$$\delta(x_1 - x_2 + x_3 - x_4) = \int \frac{dp'}{(2\pi)^4} e^{ip'(x_1 - x_2 + x_3 - x_4)} = \int \frac{dp}{(2\pi)^4} e^{p\sigma(x_1 - x_2 + x_3 - x_4)}. \quad (17)$$

Let  $\mathcal{G}$  be a ribbon graph. Choosing a particular root vertex  $\bar{V}$ , the parametric representation for the amplitude of  $\mathcal{G}$  is expressed in terms of  $t_\ell = \tanh \alpha_\ell / 2$

(where  $\alpha_\ell$  is the former Schwinger parameter) as

$$\mathcal{A}_G = \int_0^1 \prod_\ell dt_\ell (1 - t_\ell^2)^{\frac{D}{2}-1} \int dx dp e^{-\frac{\alpha}{2} X G X^t} \quad (18)$$

where  $D$  is the space-time dimension,  $X$  summarizes all positions and hypermomenta and  $G$  is a certain quadratic form. Calling  $x_e$  and  $p_{\bar{V}}$  the external variables and  $x_i, p_i$  the internal ones, we decompose  $G$  into an internal quadratic form  $Q$ , an external one  $M$  and a coupling part  $P$  so that

$$X = \begin{pmatrix} x_e & p_{\bar{V}} & x_i & p_i \end{pmatrix}, \quad G = \begin{pmatrix} M & P \\ P^t & Q \end{pmatrix}, \quad (19)$$

Performing the Gaussian integration over all internal variables one gets the non-commutative parametric representation:

$$\mathcal{A}_G = \int \prod_\ell dt_\ell (1 - t_\ell^2)^{\frac{D}{2}-1} \frac{1}{\sqrt{\det Q}} e^{-\frac{\alpha}{2} \begin{pmatrix} x_e & p_{\bar{V}} \end{pmatrix} [M - P Q^{-1} P^t] \begin{pmatrix} x_e \\ p_{\bar{V}} \end{pmatrix}}. \quad (20)$$

This representation leads to new polynomials  $HU_{\mathcal{G}, \bar{V}}$  and  $HV_{\mathcal{G}, \bar{V}}$  in the  $t_\ell$  ( $\ell = 1, \dots, L$ ) variables, analogs of the Symanzik polynomials  $U$  and  $V$  of the commutative case, through

$$\mathcal{A}_G(\{x_e\}, p_{\bar{V}}) = K' \int_0^1 \prod_\ell dt_\ell (1 - t_\ell^2)^{\frac{D}{2}-1} e^{-\frac{HV_{\mathcal{G}, \bar{V}}(t, x_e, p_{\bar{V}})}{HU_{\mathcal{G}, \bar{V}}(t)}}. \quad (21)$$

The main results of [8, 9] are

- The polynomial  $HU_{\mathcal{G}, \bar{V}}$  and the real part of  $HV_{\mathcal{G}, \bar{V}}$  have a *positivity property*. They are sums of monomials with positive integer coefficients, which are squares of Pfaffians with integer entries.
- Leading terms can be identified in a given ‘‘Hepp sector’’, at least for *orientable graphs*. In  $HU_{\mathcal{G}, \bar{V}}$  they correspond to hyper-trees which are the disjoint union of a tree in the direct graph and an other tree in the dual graph. Any connected graph has such hypertrees. Similarly ‘‘hyper-two-trees’’ govern the leading behavior of  $HV_{\mathcal{G}, \bar{V}}$  in any Hepp sector.

Let us relabel the  $2L$  internal positions of  $\mathcal{G}$  as  $L$  short and  $L$  long variables in the terminology of [5]. It has been shown in [8] that for the Grosse-Wulkenhaar  $\phi^{*4}$  model:

$$\begin{aligned} HU_{\mathcal{G}, \bar{V}}(t) &= \sum_{K_U = I \cup J, \ n+|K_U| \text{ odd}} s^{2g-k_{K_U}} n_{K_U}^2 \prod_{\ell \notin I} t_\ell \prod_{\ell' \in J} t_{\ell'} \\ &= \sum_{K_U} a_{K_U} \prod_{\ell} t_\ell^{u_{\ell K_U}} \equiv \sum_{K_U} HU_{K_U}. \end{aligned} \quad (22)$$

where

- $I$  is a subset of the first  $L$  indices (corresponding to the short variables) with  $|I|$  elements, and  $J$  a subset of the next  $L$  indices (corresponding to the long ones) with  $|J|$  elements.
- $B$  is the antisymmetric part of the quadratic form  $Q$  restricted to these  $2L$  short and long variables (i.e. omitting hypermomenta)
- $n_{K_U} = \text{Pf}(B_{\widehat{K_U}})$  is the Pfaffian of the antisymmetric matrix obtained from  $B$  by deleting the lines and columns in the set  $K_U = I \cup J$ .
- $k_{K_U}$  is  $|K_U| - L - F + 1$ , where  $F$  is the number of faces of the graph,  $g$  is the genus of  $\mathcal{G}$  and  $s$  is a constant.
- $a_{K_U} = s^{2g-k_{K_U}} n_{K_U}^2$ , and

$$u_{\ell K_U} = \begin{cases} 0 & \text{if } \ell \in I \text{ and } \ell \notin J \\ 1 & \text{if } (\ell \notin I \text{ and } \ell \notin J) \text{ or } (\ell \in I \text{ and } \ell \notin J) \\ 2 & \text{if } \ell \notin I \text{ and } \ell \in J \end{cases} . \quad (23)$$

The second polynomial  $HV$  has both a real part  $HV^{\mathcal{R}}$  and an imaginary part  $HV^{\mathcal{I}}$ , more difficult to write down. We need to introduce beyond  $I$  and  $J$  as above a particular line  $\tau \notin I$  which is the analog of a two-tree cut. We define  $\text{Pf}(B_{\widehat{K_V \hat{\tau}}})$  as the Pfaffian of the matrix obtained from  $B$  by deleting the lines and columns in the sets  $I$ ,  $J$  and  $\tau$ . Moreover we define  $\epsilon_{I, \tau}$  to be the signature of the permutation obtained from  $(1, \dots, d)$  by extracting the positions belonging to  $I$  and replacing them at the end in the order

$$1, \dots, d \rightarrow 1, \dots, \hat{i}_1, \dots, \hat{i}_{|I|}, \dots, \hat{i}_\tau, \dots, d, i_\tau, i_{|I|}, \dots, i_1 . \quad (24)$$

where  $d$  is the dimension of the matrix  $Q$ . Then:

$$\begin{aligned} HV_{\mathcal{G}, \bar{V}}^{\mathcal{R}} &= \sum_{K_V = I \cup J} \prod_{\ell \notin I} t_\ell \prod_{\ell' \in J} t_{\ell'} \left[ \sum_{e_1} x_{e_1} \sum_{\tau \notin K_V} P_{e_1 \tau \epsilon_{K_V \tau}} \text{Pf}(B_{\widehat{K_V \hat{\tau}}}) \right]^2 . \\ &= \sum_{K_V} s_{K_V}^{\mathcal{R}} \left( \prod_{\ell=1}^L t_\ell^{v_{\ell K_V}} \right) \equiv \sum_{K_V} HV_{K_V}^{\mathcal{R}} \end{aligned} \quad (25)$$

where

$$s_{K_V}^{\mathcal{R}} = \left( \sum_e x_e \sum_{\tau \notin K_V} P_{e \tau \epsilon_{K_V \tau}} \text{Pf}(B_{\widehat{K_V \hat{\tau}}}) \right)^2 \quad (26)$$

and  $v_{\ell K_V}$  given by the same formula as  $u_{\ell K_U}$ .

The imaginary part involves *pairs* of lines  $\tau, \tau'$  and corresponding signatures (see [8, 9] for details):

$$\begin{aligned}
HV_{\mathcal{G}, \bar{V}}^{\mathcal{I}} &= \sum_{K_V = I \cup J} \prod_{\ell \notin I} t_\ell \prod_{\ell' \in J} t_{\ell'} \\
&\quad \epsilon_{K_V} \text{Pf}(B_{\hat{K}_V}) \left[ \sum_{e_1, e_2} \left( \sum_{\tau, \tau'} P_{e_1 \tau} \epsilon_{K_V \tau \tau'} \text{Pf}(B_{\hat{K}_V \hat{\tau} \hat{\tau}'} ) P_{e_2 \tau'} \right) x_{e_1} \sigma x_{e_2} \right]. \\
&= \sum_{K_V} s_{K_V}^{\mathcal{I}} \left( \prod_{\ell=1}^L t_\ell^{v_{\ell K_V}} \right) \equiv \sum_{K_V} HV_{K_V}^{\mathcal{I}}
\end{aligned} \tag{27}$$

where

$$s_{K_V}^{\mathcal{I}} = \epsilon_{K_V} \text{Pf}(B_{\hat{K}_V}) \left( \sum_{e, e'} \left( \sum_{\tau, \tau'} P_{e \tau} \epsilon_{K_V \tau \tau'} \text{Pf}(B_{\hat{K}_V \hat{\tau} \hat{\tau}'} ) P_{e' \tau'} \right) x_e \sigma x_{e'} \right). \tag{28}$$

A similar development exists for the LSZ model ([9]), although more involved. The first polynomial in this case writes

$$\begin{aligned}
HU_{\mathcal{G}, \bar{V}}(t) &= \sum_{\substack{I \subset \{1 \dots L\}, \\ n+|I| \text{ odd}}} s^{2g-k_I} n_I^2 \prod_{l \in I} \frac{1+t_l^2}{2t_l} \prod_{l' \in \{1, \dots, L\}} t_{l'} \\
&= \sum_{K_U = I \cup J} a_{K_U} \prod_{\ell} t_\ell^{u_{\ell K_U}} \equiv \sum_{K_U} HU_{K_U}.
\end{aligned} \tag{29}$$

Again  $I$  runs over the subsets of  $\{1, \dots, L\}$ . But  $J$  now runs over subsets of  $I$  (representing the expansion of  $\prod_{\ell \in I} [1 + t_\ell^2]$ ), and one has again

$$u_{\ell K_U} = \begin{cases} 0 & \text{if } \ell \in I \text{ and } \ell \in J \\ 1 & \text{if } \ell \notin I \\ 2 & \text{if } \ell \in I \text{ and } \ell \notin J \end{cases}, \quad a_{K_U} = \frac{1}{2^{|I|}} s^{2g-k_I} n_I^2. \tag{30}$$

There are similar expressions for the second polynomial.

To summarize, the main differences of the NC parametric representation with respect to the commutative case are:

- the presence of the constants  $a_j$  in the form (22) of  $HU$ ,
- the presence of an imaginary part  $i HV^{\mathcal{I}}$  in  $HV$ , see (27),
- the fact that the parameters  $u_{\ell j}$  and  $v_{\ell k}$  in the formulas above can have also the value 2 (and not only 0 or 1).



## 4 Non-commutative CM representation

For the real part  $HV^{\mathcal{R}}$  of  $HV$  one uses again the identity (8) as

$$e^{-HV_{K_V}^{\mathcal{R}}/U} = \int_{\tau_{K_V}^{\mathcal{R}}} \Gamma(-y_{K_V}^{\mathcal{R}}) \left( \frac{HV_{K_V}^{\mathcal{R}}}{U} \right)^{y_{K_V}^{\mathcal{R}}}, \quad (31)$$

which introduces the set of Mellin parameters  $y_{K_V}^{\mathcal{R}}$ .

However for the imaginary part one cannot apply anymore the same identity. Nevertheless it remains true **in the sense of distributions**. More precisely we have for  $HV_{K_V}^{\mathcal{I}}/U > 0$  and  $-1 < \tau_{K_V}^{\mathcal{I}} < 0$

$$e^{-iHV_{K_V}^{\mathcal{I}}/U} = \int_{\tau_{K_V}^{\mathcal{I}}} \Gamma(-y_{K_V}^{\mathcal{I}}) \left( \frac{iHV_{K_V}^{\mathcal{I}}}{U} \right)^{y_{K_V}^{\mathcal{I}}}. \quad (32)$$

The proof is given in Appendix B.

Note that this introduces another set of Mellin parameters  $y_{K_V}^{\mathcal{I}}$ . The distributional sense of formula (32) is a translation of the distributional form of a vertex contribution, and the major difference with respect to the commutative case.

For the polynomial  $HU$  one can use again the formula (10) which rewrites here as

$$\Gamma\left(\sum_{K_V} y_{K_V} + \frac{D}{2}\right) (HU)^{-\sum_{K_V} (y_{K_V}^{\mathcal{R}} + y_{K_V}^{\mathcal{I}}) - \frac{D}{2}} = \int_{\sigma} \prod_{K_U} \Gamma(-x_{K_U}) U_{K_U}^{x_{K_U}}, \quad (33)$$

Note that the rôle of the Mellin parameters  $y_{K_V}$  of the commutative case is now played by the sum  $y_{K_V}^{\mathcal{R}} + y_{K_V}^{\mathcal{I}}$ .

As in the commutative case, we now insert the distribution formulas (31), (32) and (33) in the general form of the Feynman amplitude. This gives

$$\begin{aligned} \mathcal{A}_G &= K' \int_{\Delta} \frac{\prod_{K_U} a_{K_U}^{x_{K_U}} \Gamma(-x_{K_U})}{\Gamma(-\sum_{K_U} x_{K_U})} \left( \prod_{K_V} (s_{K_V}^{\mathcal{R}})^{y_{K_V}^{\mathcal{R}}} \Gamma(-y_{K_V}^{\mathcal{R}}) \right) \\ &\quad \left( \prod_{K_V} (s_{K_V}^{\mathcal{I}})^{y_{K_V}^{\mathcal{I}}} \Gamma(-y_{K_V}^{\mathcal{I}}) \right) \int_0^1 \prod_{\ell=1}^L dt_{\ell} (1 - t_{\ell}^2)^{\frac{D}{2}-1} t_{\ell}^{\phi_{\ell}-1} \end{aligned} \quad (34)$$

where

$$\phi_{\ell} \equiv \sum_{K_U} u_{\ell K_U} x_{K_U} + \sum_{K_V} (v_{\ell K_V}^{\mathcal{R}} y_{K_V}^{\mathcal{R}} + v_{\ell K_V}^{\mathcal{I}} y_{K_V}^{\mathcal{I}}) + 1. \quad (35)$$

Here  $\int_{\Delta}$  means integration over the variables  $\frac{\text{Im } x_{K_U}}{2\pi i}$ ,  $\frac{\text{Im } y_{K_V}^{\mathcal{R}}}{2\pi i}$  and  $\frac{\text{Im } y_{K_V}^{\mathcal{I}}}{2\pi i}$ , where  $\Delta$  is the convex domain:

$$\Delta = \left\{ \sigma, \tau^{\mathcal{R}}, \tau^{\mathcal{I}} \left| \begin{array}{l} \sigma_{K_U} < 0; \tau_{K_V}^{\mathcal{R}} < 0; -1 < \tau_{K_V}^{\mathcal{I}} < 0; \\ \sum_{K_U} x_{K_U} + \sum_{K_V} (y_{K_V}^{\mathcal{R}} + y_{K_V}^{\mathcal{I}}) = -\frac{D}{2}; \\ \forall \ell, \text{Re } \phi_{\ell} \equiv \sum_{K_U} u_{\ell K_U} \sigma_{K_U} \\ + \sum_{K_V} (v_{\ell K_V}^{\mathcal{R}} \tau_{K_V}^{\mathcal{R}} + v_{\ell K_V}^{\mathcal{I}} \tau_{K_V}^{\mathcal{I}}) + 1 > 0 \end{array} \right. \right\} \quad (36)$$

and  $\sigma$ ,  $\tau^{\mathcal{R}}$  and  $\tau^{\mathcal{I}}$  stand for  $\text{Re } x_{K_U}$ ,  $\text{Re } y_{K_V}^{\mathcal{R}}$  and  $\text{Re } y_{K_V}^{\mathcal{I}}$ .

The  $dt_\ell$  integrations in (34) may be performed using the representation for the beta function

$$\int_0^1 \prod_{\ell=1}^L dt_\ell (1 - t_\ell^2)^{\frac{D}{2}-1} t_\ell^{\phi_\ell-1} = \frac{1}{2} \beta\left(\frac{\phi_\ell}{2}, \frac{D}{2}\right). \quad (37)$$

Furthermore one has

$$\beta\left(\frac{\phi_\ell}{2}, \frac{D}{2}\right) = \frac{\Gamma(\frac{\phi_\ell}{2})\Gamma(\frac{D}{2})}{\Gamma(\frac{\phi_\ell+D}{2})}.$$

This representation is convergent for  $0 < \Re D < 2$  and we get:

**Theorem 4.1.** *Any Feynman amplitude of a  $\phi^{*4}$  graph is analytic at least in the strip  $0 < \Re D < 2$  where it admits the following CM representation*

$$\begin{aligned} \mathcal{A}_{\mathcal{G}} = & K' \int_{\Delta} \frac{\prod_{K_U} a_{K_U}^{x_{K_U}} \Gamma(-x_{K_U})}{\Gamma(-\sum_{K_U} x_{K_U})} \left( \prod_{K_V} (s_{K_V}^{\mathcal{R}})^{y_{K_V}^{\mathcal{R}}} \Gamma(-y_{K_V}^{\mathcal{R}}) \right) \\ & \left( \prod_{K_V} (s_{K_V}^{\mathcal{I}})^{y_{K_V}^{\mathcal{I}}} \Gamma(-y_{K_V}^{\mathcal{I}}) \right) \left( \prod_{\ell=1}^L \frac{\Gamma(\frac{\phi_\ell}{2})\Gamma(\frac{D}{2})}{2\Gamma(\frac{\phi_\ell+D}{2})} \right). \end{aligned} \quad (38)$$

which holds as tempered distribution of the external invariants.

**Proof** We have to check that for any  $\phi^{*4}$  graph, the domain  $\Delta$  is non empty for  $0 < \Re D < 2$ . It is obvious to check that  $\Delta$  is non empty for  $0 < \Re D < 1$ ; indeed since the integers  $u$  and  $v$  are bounded by 2, we can put an arbitrary single  $x_{K_U}$  close to  $-D/2$  and the others with negative real parts close to 0 and the conditions  $\phi_\ell > 0$  will be satisfied. With a little extra care, one also gets easily that  $\Delta$  is non empty for  $0 < \Re D < 2$ . Indeed one can put again one single  $x_{K_U}$  close to  $-D/2$ , provided for any  $\ell$  we have  $u_{\ell, K_U} \neq 2$ . For the ordinary Grosse-Wulkenhaar model one can always find  $K_U = I \cup J$  where  $I = \{1, \dots, L\}$  is the full set of lines [8], hence  $u_{\ell, K_U} \neq 2$  for any  $\ell$  by (23). In the “covariant” case treated in [9], one can simply take  $K_U$  with  $J = I$ . Considering (30) ensures again obviously that  $u_{\ell, K_U} \neq 2$  for any  $\ell$ .

The analyticity statement is part of a more precise meromorphy statement proved in the next section.  $\square$

We have thus obtained the **complete Mellin representation of Feynman amplitudes for non-commutative QFT**. Let us point out first that the distributional aspect does not affect the last product, the one involving the dimensionality of space-time  $D$ . Therefore, one can still prove the meromorphy of the Feynman amplitude  $\mathcal{A}_{\mathcal{G}}(D)$  in the space-time dimension  $D$  essentially as in the commutative case. This is done in the next section.

Another difference is that the  $dt_\ell$  integration brought  $\beta$  functions (see (37)), hence more complicated ratios of  $\Gamma$  functions than in the commutative case. In

particular singularities appear for negative  $D$  which come from the integration near  $t_\ell = 1$ , hence the “infrared side”. They have absolutely no analog in the commutative case and are due to the hyperbolic nature of the Mehler kernel. At zero mass these singularities occur for  $D$  real negative even integer and in the massive case they occur for  $D$  real but sufficiently negative depending on the mass.

## 5 Meromorphy in the space-time dimension $D$

The complete Mellin representation (38) for non-commutative theories allows a quick proof of the meromorphy of the Feynman amplitude  $\mathcal{A}_G$  in the variable  $D$ , since  $D$  appears only as argument of  $\Gamma$  functions which are meromorphic.

**Theorem 5.1.** *Any amplitude  $\mathcal{A}_G$  is a tempered meromorphic distribution in  $D$ , that is  $\mathcal{A}_G$  smeared against any fixed Schwarz-class test function of the external invariants yields a meromorphic function in  $D$  in the entire complex plane, with singularities located among a discrete rational set  $S_G$  which depends only on the graph  $G$  not on the test function. In the  $\phi^4$  case, no such singularity can occur in the strip  $0 < \Re D < 2$ , a region which is therefore a germ of analyticity common to the whole theory.*

**Proof**  $\Gamma$  functions and their inverse are meromorphic in the entire complex plane. Formula (38) is a tempered distribution which is the Fourier transform of a slow-growth function, see Appendix A. It must be smeared against a Schwarz-class test function and by definition the result can be computed in Fourier space, as an ordinary integral of the product of the slow-growth function by the rapid decaying Fourier transform of the test function. At this stage the  $D$ -dependence occurs in fact in the integrand of an ordinary integral.

We can now copy the analysis of [22, 23]. We introduce general notations  $z_\nu$  for the variables  $x$  and  $y$  and the lattices of polar varieties  $\phi_r = -n_r$  where the different  $\Gamma$  functions have their singularities. This defines convex cells, and singularities in  $D$  can occur only when the hyperplane  $P(D) = \{\sum_\nu z_\nu = -D/2\}$  crosses the vertices of that lattice. Because the coefficients  $u$  and  $v$  can only take integer values (in fact 0, 1 and 2) this give a discrete set of rational values  $S_G$  which depend on the graph, but not on the test-function. Out of these values, to check that this integral is holomorphic around any dimension which does not cross the lattice vertices is a simple consequence of commuting the Cauchy-Riemann  $\partial_{\overline{D}}$  operator with the integral through Lebesgue’s dominated convergence theorem.

Finally to check meromorphy, fix a particular  $D_0 \in S_G$ . We can write the Laurent series expansion of the integrand near that  $D_0$  which starts as  $a_{n_0}(D - D_0)^{-n_0}$ . Multiplying by  $(D - D_0)^{n_0}$  we get a finite limit as  $D \rightarrow D_0$ , hence the isolated singularity at  $D_0$  cannot be essential. Repeating the argument at each point of  $S_G$  we obtain meromorphy in the space time dimension  $D$  in the whole complex plane  $\mathbb{C}$ , apart from some discrete rational set where poles of finite order can occur.  $\square$

Recall that this theorem is the starting point for the dimensional renormalization of NCQFTs which will be published elsewhere [26].

### Appendix A: Proof of the formula (32)

Let us return to formula (32). It follows from

**Lemma 5.1.** *For  $-1 < s < 0$ , as tempered distribution in  $u$*

$$e^{-iu}\chi(u) = \chi(u) \int_{-\infty}^{+\infty} \Gamma(-s-it) (iu)^{s+it} \frac{dt}{2\pi}. \quad (39)$$

where  $\chi$  is the Heaviside step function ensuring  $u > 0$ .

**Proof** The left hand side is a locally integrable function bounded by one (together with all its derivatives) times the Heaviside function, so it is a tempered distribution. The right hand side is in fact a Fourier transform in  $t$  but taken at a value  $\log u$ . To check that this is the same tempered distribution as  $e^{-iu}\chi(u)$ , let us apply it to a Schwartz test function  $f$ . Writing the usual representation for the  $\Gamma$  function we have to compute

$$I(f) = \frac{1}{2\pi} \int_0^\infty f(u) du \int_{-\infty}^{+\infty} (iu)^{s+it} dt \int_0^\infty x^{-s-it-1} e^{-x} dx. \quad (40)$$

But the integrand is analytic in  $x$  in the open upper right quarter of the complex plane. Let us rotate the contour from  $x$  to  $ix$ . The contribution of the quarter circle with radius sent to infinity tends to zero because of the condition  $-1 < s < 0$ , and we find

$$\begin{aligned} I(f) &= \frac{1}{2\pi} \int_0^\infty f(u) du \int_{-\infty}^{+\infty} (iu)^{s+it} dt \lim_{R \rightarrow \infty} \int_0^R (ix)^{-s-it-1} e^{-ix} i dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt \int_{-\infty}^\infty f(e^v) e^{v(s+1)} e^{itv} dv \lim_{R \rightarrow \infty} \int_0^R x^{-s-it-1} e^{-ix} dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt \hat{G}(t) \lim_{R \rightarrow \infty} \int_0^R x^{-s-it-1} e^{-ix} dx \\ &= \lim_{R \rightarrow \infty} \int_0^R dx e^{-ix} x^{-s-1} \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt \hat{G}(t) e^{-it \log x} \\ &= \int_0^\infty f(x) e^{-ix} dx. \end{aligned} \quad (41)$$

where  $G(v) = f(e^v) e^{v(s+1)}$  is still rapid decay and in the second line we change variables from  $u$  to  $v = \log u$  and use the definition of Fourier transform for distributions.  $\square$

### Appendix B: The massive case

In the massive case we have to modify equation (37) to include the additional  $e^{-\alpha_\ell m^2}$  per line, but unfortunately we have  $e^{-\alpha_\ell} = \frac{1-t_\ell}{1+t_\ell}$  so that (37) becomes

$$\int_0^1 \prod_{\ell=1}^L dt_\ell (1-t_\ell^2)^{\frac{D}{2}-1} \left( \frac{1-t_\ell}{1+t_\ell} \right)^{m^2} t_\ell^{\phi_\ell-1} = \frac{1}{2} \beta_m \left( \frac{\phi_\ell}{2}, \frac{D}{2} \right). \quad (42)$$

where the modified “ $\beta_m$ ” function is no longer an explicit quotient of  $\Gamma$  functions, and depends of the mass  $m$ .

It is rather easy to check that the analyticity and meromorphy properties of this modified beta function are similar to those of the ordinary beta function, at least for  $\Re D > 0$ . Indeed

$$\beta_m\left(\frac{\phi_\ell}{2}, \frac{D}{2}\right) - \beta\left(\frac{\phi_\ell}{2}, \frac{D}{2}\right) = 2 \int_0^1 \prod_{\ell=1}^L dt_\ell (1 - t_\ell^2)^{\frac{D}{2}-1} \left[ \left( \frac{1 - t_\ell}{1 + t_\ell} \right)^{m^2} - 1 \right] t_\ell^{\phi_\ell-1}. \quad (43)$$

The integral above is convergent for  $t_\ell$  close to 1 if  $\Re D > 0$  and for  $t_\ell$  close to 0 if  $\Re \phi_\ell > -1$  so that it is an analytic function of  $D$  and  $\phi_\ell$  in this domain, hence  $\beta$  and  $\beta_m$  have the same singularity structure.

Nevertheless this CM representation becomes less explicit and therefore less attractive in this massive case. Remark however that for  $\Re D > 0$ , masses are not essential to the analysis of vulcanized NC field theories which have no infrared divergencies and only “half-a-direction” for their renormalization group anyway. As remarked earlier, masses can only push further away the infrared singularities at negative  $D$  that come from the hyperbolic nature of the Mehler kernel. But it is not clear whether such infrared singularities at negative non commutative dimensions have any physical interpretation.

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